Appendix D
Third-Order Stationary Phase Theory

D.1 Maximum Flaring

At the first contact of a ray with a caustic surface, the radius of the first Fresnel zone becomes infinite. Both the first and second derivatives of the Fresnel phase function vanish at this point. In a geometric optics framework, flaring in observed signal amplitude will be a maximum (in fact, infinite) at this point [see Eqs. (2.2-5) and (2.2-6)]. In a Fresnel framework, the signal amplitude will be large in this neighborhood, but not infinite, and the point where it reaches a maximum may be slightly offset from the first contact point where geometric optics based on a second-order theory predicts infinite signal power.

The maximum and the offset can be estimated using a third-order stationary phase treatment and the thin phase screen model discussed in Chapter 2. Let us expand the Fresnel phase function $\Phi(h, h_{\text{LG}})$ defined in Eq. (2.5-1) in a Taylor series about $h^{\dagger}$, the point of zero convexity, where $\partial^2\Phi/\partial h^2 = 0$. We assume that the stationary phase points where $\partial\Phi/\partial h = 0$ are nearby; thus, the principal contributions to the Rayleigh–Sommerfeld integral for the observed signal will come from the immediate neighborhood about $h^{\dagger}$. When $h^{\dagger}$ is well away from an integration end point, the integral $I(h_{\text{LG}}) = E(h_{\text{LG}})\exp[i\psi(h_{\text{LG}})]$ in Eq. (2.5-1) becomes, to a good approximation,

$$I(h_{\text{LG}}) \doteq \sqrt{\frac{2}{\lambda D}}\frac{1}{1+i}\exp(i\Phi^{\dagger})\int_{-\infty}^{\infty}\exp\left(i\left(\Phi'^{\dagger}(h-h^{\dagger}) + \frac{1}{6}\Phi'''^{\dagger}(h-h^{\dagger})^3\right)\right)dh \quad \text{(D-1)}$$

From the stationary phase and thin-screen methodology in Chapter 2, we have

$$\left.\begin{aligned}
\Phi^{\dagger} &= \frac{\pi}{\lambda D}\left(h^{\dagger}-h_{\mathrm{LG}}\right)^{2}+\frac{2\pi}{\lambda}\int_{h^{\dagger}}^{\infty}\alpha(h)dh \\
\Phi'^{\dagger} &= \left.\frac{\partial\Phi}{\partial h}\right|_{h^{\dagger}} = \frac{2\pi}{\lambda D}\left[h^{\dagger}-h_{\mathrm{LG}}-D\alpha\left(h^{\dagger}\right)\right] \\
\Phi''^{\dagger} &= \left.\frac{\partial^{2}\Phi}{\partial h^{2}}\right|_{h^{\dagger}} = \frac{2\pi}{\lambda D}\left(1-D\alpha'\left(h^{\dagger}\right)\right)=0 \\
\Phi'''^{\dagger} &= \left.\frac{\partial^{3}\Phi}{\partial h^{3}}\right|_{h^{\dagger}} = -\frac{2\pi}{\lambda}\alpha''\left(h^{\dagger}\right)
\end{aligned}\right\} \tag{D-2}$$

The Fresnel phase and its partial derivatives are evaluated at the zero convexity point in *h*-space. We note that $\Phi^{\dagger}$ and $\Phi'^{\dagger}$ vary with h_{LG} and, therefore, with time. However, $\Phi''^{\dagger}$ (and, therefore, $h^{\dagger}$) and also $\Phi'''^{\dagger}$ are independent of h_{LG}. We also note in passing that a zero convexity point requires that the radial gradient of the bending angle be positive, which is of course the same necessary (but not sufficient) condition for the existence of a caustic. The integral in Eq. (D-1) can be evaluated in terms of the Airy function of the first kind [1–3]. We make a change of integration variable in Eq. (D-1), $\Phi'''^{\dagger}\left(h-h^{\dagger}\right)^{3}/6=z^{3}$, to obtain for the signal amplitude

$$\left|I\left(h_{\mathrm{LG}}\right)\right| \doteq \sqrt{\frac{1}{\lambda D}}\left(\frac{6}{\left|\Phi'''^{\dagger}\right|}\right)^{1/3} J\left[a\left(h_{\mathrm{LG}}\right)\right] \tag{D-3}$$

where the function $J\left[a\left(h_{\mathrm{LG}}\right)\right]$ is given by

$$J(a)=\int_{-\infty}^{\infty}\exp\left(i\left(az+z^{3}\right)\right)dz = 2\pi 3^{-1/3}\,\mathrm{Ai}\left[a3^{-1/3}\right] \tag{D-4}$$

where $\mathrm{Ai}[y]$ is the Airy function of the first kind. The quantity *a* is given by

$$a\left(h_{\mathrm{LG}}\right)=\Phi'^{\dagger}\left(\frac{6}{\left|\Phi'''^{\dagger}\right|}\right)^{1/3} = \frac{2\pi}{\lambda D}\left[h^{\dagger}-h_{\mathrm{LG}}-D\alpha\left(h^{\dagger}\right)\right]\left(\frac{3\lambda}{\pi\left|\alpha''\left(h^{\dagger}\right)\right|}\right)^{1/3} \tag{D-5}$$

The function $J(x)$ in Eq. (D-4) is generated by the differential equation $d^{2}J/dx^{2}=xJ/3$. The solutions are oscillatory for negative x, and, for the case where the boundary values are set by the exact numerical values of $J(0)$ and $J'(0)$ from Eq. (D-4), $J(x)$ damps to zero exponentially with increasing positive x. In addition to diffraction problems, Airy functions arise in classical electrodynamics in connection with the spectral properties of synchrotron radiation, and also in quantum mechanical potential well problems where the positive z-regime corresponds to quantum tunneling processes. Airy functions can be solved in terms of certain Bessel functions of fractional order $\pm 1/3$, which are tabulated. Airy functions of the first and second kind (the latter grows exponentially large for increasing positive x) provide the asymptotic forms for the Bessel functions for large spectral number and argument.

Figure D-1 shows the behavior of $J(a)$ as a function of the parameter a around zero, including the applicable asymptotic form for the positive regime. The oscillatory behavior of the Airy function $\mathrm{Ai}[y]$ for negative y is shown in [1]. Positive real values of a correspond to that range of h_{LG} values where no real stationary phase points exist near $h^{\dagger}$. For negative values of a, two stationary phase points for $\Phi(h, h_{\mathrm{LG}})$ exist at $h - h^{\dagger} = \pm\sqrt{2\Phi'^{\dagger} / |\Phi'''^{\dagger}|}$. The oscillatory behavior of $J(a)$ for the negative regime arises from the phase interference between the contributions to Eq. (D-1) from the neighborhoods around those two points.

A caustic contact point occurs when both $\Phi''^{\dagger}$ and $\Phi'^{\dagger}$ (and a) are zero. At this point, $J(0) = 2\pi / (3\Gamma[2/3]) = 1.547$. As time varies during an occultation, h_{LG} varies nearly linearly and, consequently, so also does a. From Eq. (D-4), setting $dJ/da = 0$ yields the point of maximum flaring, which occurs at $a = -1.469$; the maximum value of $J\left[a\left(h_{\mathrm{LG}}\right)\right]$ is 2.334. Hence,

$$\left|I\left(h_{\mathrm{LG}}\right)\right|_{\mathrm{MAX}} = \left|I\left(\hat{h}_{\mathrm{LG}}\right)\right| \doteq 2.33\sqrt{\frac{1}{\lambda D}}\left(\frac{6}{\left|\Phi'''^{\dagger}\right|}\right)^{1/3} \tag{D-6}$$

and

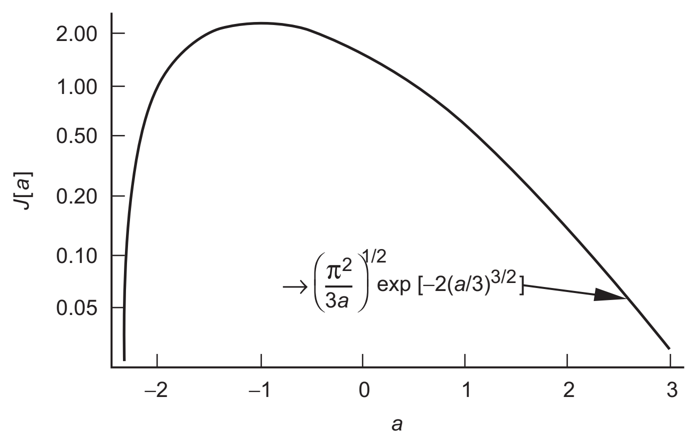

Fig. D-1. First lobe of the Airy function of the first kind.

$$\left.\begin{aligned} \hat{h}_{\mathrm{LG}} - h_{\mathrm{LG}}^{\dagger} &= \hat{h}_{\mathrm{LG}} - \left(h^{\dagger} - D\alpha\left(h^{\dagger}\right)\right) \\ &= 1.47\frac{\lambda D}{2\pi}\left(\frac{\left|\Phi'''^{\dagger}\right|}{6}\right)^{1/3} = 1.47D\left(\frac{\lambda^2\left|\alpha''\left(h^{\dagger}\right)\right|}{24\pi^2}\right)^{1/3} \\ \alpha'\left(h^{\dagger}\right) &= \frac{1}{D} \end{aligned}\right\} \tag{D-7}$$

Thus, maximum flaring occurs at $\hat{h}_{\mathrm{LG}}$, which is offset from the first contact with the caustic at $h_{\mathrm{LG}}^{\dagger}$, where $h_{\mathrm{LG}}^{\dagger} = h^{\dagger} - D\alpha(h^{\dagger}) = h^{\dagger} - \alpha(h^{\dagger})/\alpha'(h^{\dagger})$. For example, the first contact point for the upper caustic in Fig. 2-25 (i.e., the one where $h_{\mathrm{LG}} - h_o > 0$) is predicted using the model in Eq. (2.8-33) to be located at $h_{\mathrm{LG}}^{\dagger} - h_o = 600\ \mathrm{m}$, but Eq. (D-7) predicts an additional third-order offset of $\hat{h}_{\mathrm{LG}} - (h^{\dagger} - D\alpha(h^{\dagger})) = 278\ \mathrm{m}$ for maximum flaring, or a total of 878 m above the boundary. The exact value of the location of the maximum flaring point above the boundary is +846 m, which is the location shown in Fig. 2-25. This is good agreement, considering that Eq. (D-1) is a truncated version of the complete convolution integral given by Eq. (2.5-1). The value of $|I|_{\mathrm{Max}}$ predicted by Eq. (D-6) for this example is 1.52, which agrees with the exact result shown in Fig. 2-25 to better than 1 percent.

This offset of the local maximum in intensity from the position of the caustic has its analog in the theory of the rainbow. In 1838, George Airy first demonstrated using ray optics that the scattering angle of any given color in the rainbow, which is a caustic phenomenon, is similarly offset. The Airy function of the first kind originates from his study of this problem [3].

D.2 Minimum Signal Amplitude in a Shadow Zone

A fade-out in signal amplitude occurs if the stationary phase points of the Fresnel phase function $\Phi\left(h, h_{\mathrm{LG}}\right)$ are located in neighborhoods of very large convexity: $|\partial^2\Phi/\partial h^2| = |\left(2\pi/\lambda D\right)\left(1 - D\,d\alpha/dh\right)| >> 1$, so that only a small contribution to the convolution integral in Eq. (2.5-1) is obtained from such points. Thus, fade-outs are associated with very large gradients in the bending angle of a certain polarity. In the limiting case, $\Phi\left(h, h_{\mathrm{LG}}\right)$ has no stationary values anywhere in h-space within the integration limits of Eq. (2.5-1). Since $\Phi\left(h, h_{\mathrm{LG}}\right)$ must grow infinite with increasing $|h|$, it follows for this extreme case that $\partial\Phi/\partial h$ must be discontinuous at some point, and so also must be the bending angle. Two examples of such behavior in the Fresnel phase function are shown in Fig. D-2. For this situation, the principal contributions to the integral in Eq. (2.5-1) will come from those neighborhoods in h-space where

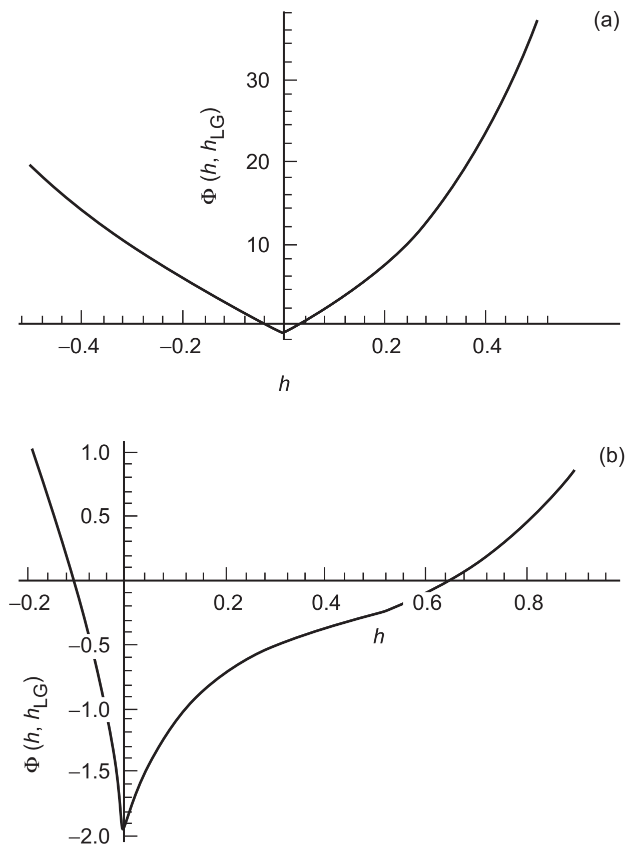

Fig. D-2. Examples of Fresnel phase function convexity for a discontinuous refractivity: (a) positive convexity throughout and (b) positive and negative convexity.

$|\partial\Phi/\partial h|$ is a minimum, which will occur either at the cusp when the convexity is positive throughout, as shown in Fig. D-2(a), or at one or more interior points where $\partial^2\Phi/\partial h^2 = 0$, an example of which is shown in Fig. D-2(b). It is possible for $\Phi(h, h_{\mathrm{LG}})$ to have a proper stationary phase point, but with a convexity so large there that the principal contribution to the convolution integral comes from some other neighborhood where $\partial\Phi/\partial h$ is small but not zero. An example is obtained from Fig. D-2(b) by simply "rounding" the cusp point. Geometric optics fails when this type of condition applies.

Let us consider the simplest case of a single cusp and positive convexity throughout, as shown by Fig. D-2(a). In this case, most of the contribution to the Fresnel integral comes from the neighborhood around the cusp. Expanding the Fresnel phase function in a Taylor series through second order about the cusp position h_o, and using the stationary phase approximation technique, Eq. (2.5-1) becomes

$$I(h_{\mathrm{LG}}) \doteq \sqrt{\frac{2}{\lambda D}}\frac{e^{i\Phi_o}}{1+i}\left(\int_{-\infty}^{0} e^{i\left(\frac{\partial\Phi_o^-}{\partial x}x+\frac{1}{2}\frac{\partial^2\Phi_o^-}{\partial x^2}x^2\right)}dx+\int_0^{\infty} e^{i\left(\frac{\partial\Phi_o^+}{\partial x}x+\frac{1}{2}\frac{\partial^2\Phi_o^+}{\partial x^2}x^2\right)}dx\right) \quad \text{(D-8)}$$

where $x = h - h_o$. The superscripts "+" and "–" on the derivatives of Φ denote, respectively, the Fresnel phase functions applicable to each regime, which are evaluated just above (for +) and just below (for –) the cusp position at $x = h - h_o = 0$. Because we are dealing with a trough in signal-to-noise ratio (SNR), the magnitude of the ratio $\partial\Phi_o / \partial x \big/ \partial^2\Phi_o / \partial x^2$ is necessarily large, and the ratio itself will be negative in the "–" regime and positive in the "+" regime. Therefore, we can use an asymptotic expansion for each Fresnel integral. After completing the square and retaining the leading term in the asymptotic expansion, Eq. (D-8) becomes

$$I(h_{\mathrm{LG}}) \doteq \frac{1}{1+i}\sqrt{\frac{2}{\lambda D}}\frac{e^{i\Phi_o}}{i\pi}\left[\left(\frac{\partial\Phi^-}{\partial x}\right)^{-1} - \left(\frac{\partial\Phi^+}{\partial x}\right)^{-1}\right]\Bigg|_{x=0} \quad \text{(D-9)}$$

If we use the thin-screen relation $\partial\Phi_o^{\pm} / \partial h = (2\pi / \lambda D)\left[h_o - h_{\mathrm{LG}} - D\alpha^{\pm}(h_o)\right]$, take the absolute value of Eq. (D-9), and minimize $\left|I(h_{\mathrm{LG}})\right|$ with respect to h_{LG}, we obtain a minimum at

$$h_o - h_{\mathrm{LG}}^{\dagger} = \frac{D}{2}\left(\alpha^-(h_o) + \alpha^+(h_o)\right) \quad \text{(D-10)}$$

The minimum value is given by

$$|I|_{\mathrm{Min}} \doteq \frac{2}{\pi}\sqrt{\frac{\lambda}{D}}\left[\frac{1}{\alpha^+(h_o) - \alpha^-(h_o)}\right] \quad \text{(D-11)}$$

For a class of discontinuity as shown in Fig. D-2(a), the minimum of the trough in SNR will be inversely proportional to the discontinuity in bending angle at the boundary at $h = h_o$. This form for the darkening in the discontinuous case should be compared with the case where $\Phi(h, h_{\mathrm{LG}})$ has a single stationary phase point in h-space (the position of which is a function of h_{LG}) but where $\Phi(h, h_{\mathrm{LG}})$ has a very large convexity and the convexity is positive throughout the range of integration in h-space. For this case, we obtain from the stationary phase technique the form

$$\left.\begin{aligned} |I|_{\text{Min}} &= \left|I\left(h_{\text{LG}}^{\dagger}\right)\right| \doteq \min_{h_{\text{LG}}}\left[\sqrt{\xi\left[h\left(h_{\text{LG}}\right)\right]}\right] = \min_{h_{\text{LG}}}\left[\left|1 - D\frac{d\alpha}{dh}\right|_{h(h_{\text{LG}})}^{-(1/2)}\right] \\ & h\left(h_{\text{LG}}\right) - h_{\text{LG}} - D\alpha\left[h\left(h_{\text{LG}}\right)\right] = 0 \end{aligned}\right\} \tag{D-12}$$

For a discontinuity as shown in Fig. D-2(b), where the principal part of the contribution to the Fresnel integration comes from interior point(s) away from the cusp, we will need to expand the Fresnel phase function about the points where $\partial^2\Phi / \partial h^2 = 0$, as given in Eq. (D-1). As an example, let us assume that there is only one zero convexity point in each regime located at h^+ in the "+" regime and at h^- in the "–" regime, which is taken to be below the "+" regime. From the thin-screen model, h^+ and h^- are defined respectively by the conditions

$$\left.\begin{aligned} 1 - D\frac{d\alpha^+}{dh}\bigg|_{h=h^+} &= 0 \\ 1 - D\frac{d\alpha^-}{dh}\bigg|_{h=h^-} &= 0 \end{aligned}\right\} \tag{D-13}$$

For this case,

$$\begin{aligned} |I(h_{\text{LG}})|^2 = \frac{1}{\lambda D}\Bigg[&\left(\frac{6}{\Phi^{+'''}}\right)^{\frac{2}{3}} J^2\left[a^+\left(h_{\text{LG}}\right)\right] + \left(\frac{6}{\Phi^{-'''}}\right)^{\frac{2}{3}} J^2\left[a^-\left(h_{\text{LG}}\right)\right] \\ &+ 2\left(\frac{6}{\Phi^{+'''}}\right)^{\frac{1}{3}}\left(\frac{6}{\Phi^{-'''}}\right)^{\frac{1}{3}} J\left[a^+\left(h_{\text{LG}}\right)\right] J\left[a^-\left(h_{\text{LG}}\right)\right]\cos(\Phi^+ - \Phi^-)\Bigg] \end{aligned} \tag{D-14}$$

where $a^{\pm}\left(h_{\text{LG}}\right)$ is defined in Eq. (D-5) for each regime. From the thin-screen model, Φ^+ and Φ^- are given by

$$k^{-1}\Phi^{\pm} = \begin{cases} \dfrac{\left(h^+ - h_{\text{LG}}\right)^2}{2D} + \displaystyle\int_{h_o}^{\infty} \alpha^+(h')dh', & \text{"+" regime} \\ \dfrac{\left(h^- - h_{\text{LG}}\right)^2}{2D} + \displaystyle\int_{h_o}^{\infty} \alpha^+(h')dh' + h\int_{r}^{h_o} \alpha^-(h')dh', & \text{"–" regime} \end{cases} \tag{D-15}$$

One needs to find the minimum of $\left|I\left(h_{\text{LG}}\right)\right|$ in Eq. (D-14) with respect to h_{LG}, but the formal solution is tedious. It is easier to work with specific models in hand. Note also the interference arising from the $\cos(\Phi^+ - \Phi^-)$ term, which varies with $h_{\rm LG}$. Equation (D-14) may be considered as providing the magnitude of the vector addition of the Fresnel integration vector component from each regime. $\left|I(h_{\rm LG})\right|$ will be minimized at that value of $h_{\rm LG}$, where these two vectors maximally cancel each other upon addition.

For the ionospheric model used in Eq. (2.8-2), for which the Fresnel effects are shown in Figs. 2-17, 2-18, and 2-19, the Fresnel phase function in the trough of Fig. 2-19 (at $h_o - h_{\rm LG} =\sim 600\ {\rm m}$) is shown schematically in Fig. D-2(b). Here only one zero convexity point exists (for $h_{\rm LG}$ in the vicinity of the trough). Thus, in this case the minimum of the SNR trough involves a trade-off between the contribution to the integration from the zero convexity neighborhood in the "–" regime and the contribution from the "+" regime at the boundary. For this specific model, Eq. (D-14) becomes

$$\left|I(h_{\rm LG})\right|^2 =$$
$$\frac{1}{\lambda D}\left[\left(\frac{1}{\Phi_o'}\right)^2 + \left(\frac{6}{\Phi^{-'''}}\right)^{\frac{2}{3}} J^2\left[a^-(h_{\rm LG})\right] + 2\left(\frac{1}{\Phi_o'}\right)\left(\frac{6}{\Phi^{-'''}}\right)^{\frac{1}{3}} J\left[a^-(h_{\rm LG})\right]\cos\left[\Phi_o - \Phi^-\right]\right] \tag{D-16}$$

where, for $h_o - h\_ >> -r_o\Delta N$, we have from Eqs. (2.5.1) and (2.8-3)

$$\left.\begin{aligned}
\Phi_o' &\doteq \frac{2\pi}{\lambda D}\left(h_o - h_{\rm LG}\right) \\
\Phi^{-'} &\doteq \frac{2\pi}{\lambda D}\left(h_o - h_{\rm LG} + r_o\varpi^{1/3}\right) \\
\Phi^{-'''} &\doteq \frac{3\pi\Delta N}{\sqrt{2\lambda r_o^2}}\varpi^{5/6}
\end{aligned}\right\} \tag{D-17}$$

where

$$\left.\begin{aligned}
\Phi_o &\doteq \frac{\pi}{\lambda D}\left(h_o - h_{\rm LG}\right)^2, \\
\Phi^- &\doteq \frac{\pi}{\lambda D}\left(h_o - h_{\rm LG} + r_o\varpi^{1/3}\right)^2 + \frac{4\pi r_o}{3\lambda}\left(-(-2\Delta N)^{3/2} - 3\Delta N\varpi^{1/6}\right)
\end{aligned}\right\} \tag{D-18}$$

and where

$$\varpi = \frac{D^2(\Delta N)^2}{2r_o^2} \tag{D-19}$$

Finding the minimum value of $\left|I(h_{\mathrm{LG}})\right|$ with respect to h_{LG} from Eq. (D-16) using these constraining relations in Eqs. (D-4), (D-17), and (D-18) is straightforward, but in view of the complexity of the stationary phase technique in this case, a straight integration of Eq. (2.5-1) seems simpler.

D.3 Accuracy of the Stationary Phase Technique

The stationary phase technique for integrating Eq. (2.5-1) traditionally has been applied only at stationary points of the Fresnel phase function $\Phi(h, h_{\mathrm{LG}})$ or at integration end points. Referring to Fig. 2-9, we see a case in Fig. 2-9(a) where the technique works well, in spite of the reversals in polarity of the convexity of $\Phi(h, h_{\mathrm{LG}})$ in h-space. In Fig. 2-9(b), the technique is compromised by the additional contribution from the neighborhood around the zero convexity point (near an altitude of 10 km) where $|\partial\Phi/\partial h|$ is a minimum, but not zero. Lastly, in Fig. 2-9(c), we see a virtually hopeless case for the stationary phase technique.

When a worrisome zero convexity point is well isolated from end points and stationary phase points, we can use Eq. (D-3) to estimate its contribution to the overall diffraction integral in Eq. (2.5-1). Let us call this contribution $|I^{\dagger}|$, which is given by Eq. (D-3) with the partial derivatives of $\Phi(h, h_{\mathrm{LG}})$ evaluated at the zero convexity point $h^{\dagger}$ where $|\partial\Phi/\partial h|$ is a local minimum.

Let $\Phi(h, h_{\mathrm{LG}})$ for a given value of h_{LG} have a stationary value at $h = h^*(h_{\mathrm{LG}})$, where $h^*(h_{\mathrm{LG}})$ is defined by $h_{\mathrm{LG}} - h^* + D\alpha(h^*) = 0$. Then, applying the stationary phase technique at h^*, one obtains for the amplitude of the signal

$$\left.\begin{aligned} \left|I^*\right| &= \left(\frac{2\pi}{\lambda D\left|\Phi^{*\prime\prime}\right|}\right)^{1/2} \\ \Phi^{*\prime\prime} &= \left.\frac{\partial^2\Phi}{\partial h^2}\right|_{h=h^*(h_{\mathrm{LG}})} \end{aligned}\right\} \tag{D-20}$$

Using Eqs. (D-3) and (D-20), the ratio $\left|I^{\dagger}\right|/\left|I^*\right|$ is given by

$$\frac{\left|I^{\dagger}\right|}{\left|I^*\right|} = \sqrt{\frac{2\left|\Phi''^*\right|}{\pi}}\left(\frac{6}{\left|\Phi'''^{\dagger}\right|}\right)^{1/3} J(a) \tag{D-21}$$

We require that $\left|I^{\dagger}\right|/\left|I^*\right| < \varepsilon$, where ε, for example, might be 1 percent. Let $a^{\dagger}$ be that value of $a > 0$ such that

$$J\left(a^{\dagger}\right) = \varepsilon \sqrt{\frac{\pi}{2\Phi''^{*}}} \left(\frac{\left|\Phi'''^{\dagger}\right|}{6}\right)^{1/3} \Bigg|_{a=a^{\dagger}} \doteq \frac{1.27}{\left(a^{\dagger}\right)^{1/4}} \exp\left(-2\left(\frac{a^{\dagger}}{3}\right)^{3/2}\right) \quad \text{(D-22)}$$

Then, from Eq. (D-5), the condition on $\Phi'^{\dagger}$ to achieve a relative accuracy of ε with the stationary phase technique applied only to the stationary phase point is given by

$$\left|\Phi'^{\dagger}\right| < a^{\dagger} \left(\frac{\left|\Phi'''^{\dagger}\right|}{6}\right)^{1/3} \quad \text{(D-23)}$$

For 1 percent accuracy, $a^{\dagger}$ typically would be in the range from 5 to 10.

References

[1] M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables,* National Bureau of Standards Series 55, Washington, DC, 1964.

[2] F. Oliver, *Introduction to Asymptotics and Special Functions*, New York: Academic Press, 1974.

[3] G. Airy, "On the Intensity of Light in the Neighborhood of a Caustic," *Trans. Cambridge Phil. Soc.*, vol. 6, pp. 379–404, 1838.